Relativistic Theory for Syntonization of Clocks in the Vicinity of the Earth

P. Wolf[1,2] and G. Petit[1]
[1]Bureau International des Poids et Mesures,
Pavillon de Breteuil, 92312 Sèvres CEDEX, France
[2]Queen Mary and Westfield College, School of Mathematical Sciences,
Mile End Road, London E1 4NS, Great Britain

Abstract

A well known prediction of Einstein's general theory of relativity states that two ideal clocks that move with a relative velocity, and are submitted to different gravitational fields will, in general, be observed to run at different rates. Similarly the rate of a clock with respect to the coordinate time of some spacetime reference system is dependent on the velocity of the clock in that reference system and on the gravitational fields it is submitted to. For the syntonization of clocks and the realization of coordinate times (like TAI) this rate shift has to be taken into account at an accuracy level which should be below the frequency stability of the clocks in question, i.e. all terms that are larger than the instability of the clocks should be corrected for. We present a theory for the calculation of the relativistic rate shift for clocks in the vicinity of the Earth, including all terms larger than one part in 10^{18}. This, together with previous work on clock synchronization (Petit & Wolf 1993, 1994), amounts to a complete relativistic theory for the realization of coordinate time scales at picosecond synchronization and 10^{-18} syntonization accuracy, which should be sufficient to accomodate future developments in time transfer and clock technology.

1. Introduction

When using the concept of syntonization in a relativistic context certain ambiguities might appear which can lead to confusion and misunderstanding. It is therefore essential to first clarify the different meanings of the expression as used in time metrology within a relativistic framework.

Consider first the case where the relative rate of two distant clocks A and B is measured directly i.e. the frequencies of two signals coming from A and B respectively are compared by some observer 0. Taking the case where the observer is in the immediate vicinity of B and at rest with respect to B the measured relative rate is predicted as;

$$(d\tau_A/d\tau_B) - O = 1 + [(U_B - U_A) - v^2/2]/c^2 + O(c^{-4}) \tag{1}$$

in the first post–Newtonian approximation where $(d\tau_A/d\tau_B)_O$ is the relative rate of the two clocks as observed by 0, U is the total gravitational potential at the location of the clock, v is the relative speed of the two clocks and c is the speed of light in vacuum. Note that this result is completely dependent on the observer 0. If, for example, 0 was in the immediate vicinity of A and at rest with respect to A the term in $v^{@}$ would change sign. Note also that (1) is independent of any reference frame or coordinate system. It is a coordinate independent, measurable quantity.

For the realization of coordinate time scales (like TAI) it is necessary to syntonize clocks with respect to the coordinate time in question, *i.e.* to determine the rate of a clock A with respect to an ideal coordinate time of some space–time reference frame. For example, using a geocentric non–rotating frame with TCG as coordinate time (as defined by the IAU /(1991)) we obtain, again in the first post–Newtonian approximation;

$$d\tau_A/dTCG = 1 - [U(\mathbf{w}) + v^2/2]/c^2 + O(c^{-4}) \tag{2}$$

where $(cTCG, w^k)$ are coordinates in the geocentric frame with w representing the triplet w^k. The potential at the position of the clock $U(\mathbf{w})$ is the sum of the Earth's potential and tidal potentials of external bodies, and $v = ((dw^i/dTCG)(dw^i/dTCG))^{1/2}$ is the coordinate speed of the clock in the geocentric, non–rotating frame. Note that this rate depends entirely on the chosen reference frame. It is a coordinate quantity which cannot be obtained directly from measurement, but must be calculated theoretically using the definition of the reference frame in question with the appropriate metric equation.

When using repeated time transfers employing the convention of coordinate synchronization (Allan & Ashby 1986, Petit & Wolf 1994) for the determination of the relative rate of two clocks A and B, the resulting rate predicted by theory is simply:

$$dTA/dTB = (dTAIdTCG)(dTCGIdTB)(3) \tag{3}$$

with $d\tau/dTCG$ given in (2). This is a combination of coordinate dependent quantities and not to be confused with the measurable quantity expressed in (1). The former is entirely dependent on the chosen reference frame and the convention of synchronization while the latter is specific to the measuring observer 0. They will, in general, differ due to, essentially, the difference in the v^2/c^2 terms. In sections 2 and 3 we will consider the syntonization of clocks with respect to coordinate times TCG (Geocentric Coordinate Time) and TT (Terrestial Time, the ideal form of TAI) as defined by the IAU (1991) together with the transformation relating the two. The aim is to provide expressions in the form of (2) including all terms whose magnitudes exceed current and near future clock stabilities which are estimated to reach parts in 10^{18}, as shown in Figure I (Maleki 1993).

When determining the relative rate of two distant clocks, one might be interested in time varying effects only (i.e. effects that influence the observed frequency stability), which, as will be shown, can be calculated at higher accuracies than constant frequency shifts. They are discussed briefly in section 5.

2. Syntonization with respect to TCG

Using the metric given in resolution A4 of the IAU (1991) the relation between the proper time of a clock T and TCG can be expressed a:

$$d\tau/dTCG = 1 - [U(\mathbf{w}) + \bar{U}(\mathbf{x}_E + \mathbf{w}) - \bar{U}(\mathbf{x}_E) - \bar{U}_{,k}(\mathbf{x}_E)w^k + v^2/2 + \mathbf{Q}_k]/C^2 + O(c^{-4}) \qquad (4)$$

where coordinates in the barycentric frame are represented by $(cTCB, x^k)$ with x denoting the triplet x^k and the subscript E referring to the Earth's center of mass. $U_E(w)$ and $\bar{U}(\mathbf{x})$ are the Newtonian gravitational potentials of the Earth and of external masses respectively, $v = ((dw^i/dTCG)(dw^i/dTCG))^{\frac{1}{2}}$, the coordinate speed of the clock in the geocentric, non-rotating frame and Q_k is the correction for the non-geodesic barycentric motion of the Earth.

We find that in the vicinity of the Earth the term in Q and terms of order C^{-4} (given explicitly in Brumberg & Kopejkin 1990 and Kopejkin 1988) amount to a few parts in 10" or less. This implies that the specification of coordinate conditions (harmonic, standard post–Newtonian etc.…) and the state of rotation of the frame (kinematically or dynamically non–rotating) is not significant for syntonization at the $10^{-1}8$ accuracy level.

All effects that need to be taken into account for the calculation of the remaining terms are listed in tables la and lb, together with orders of magnitude and present day uncertainties of the associated corrections.

Syntonization with respect to TCG of Earth–bound clocks is limited at the 10^{-17} accuracy level by uncertainties in the determination of the potential of the Earth at the location of the clock. Hence only effects whose influence on (4) is larger than this limit are considered in Table 1a.

The gravitational potential of the Earth, $U_E(w)$ can be expressed as a series expansion in spherical harmonics. However, owing to mass irregularities such a series must be considered divergent at the surface of the Earth (Moritz 1961). Nonetheless, due to the predominantly ellipsoidal shape of the Earth, one can use the first two terms of this series expansion as a first approximation (Allan & Ashby 1986, CCIR 1990, Klioner 1992). Thus:

$$U_E(\mathbf{w}) = GM_E\alpha_1^2 J_2(1 - 3\cos^2\theta)/2w^3 \cdots \qquad (5)$$

where G is the Newtonian gravitational constant, M_E is the mass of the Earth, α_1, and J_2 ($J_2 = 1.0826 \times 10^{-3}$) are the equatorial radius and the quadrupole moment coefficient of the Earth respectively and θ is the geocentric colatitude of the point of interest.

Substituting (5) into the second term of (4) gives terms of the order of $7 \times 1O^{-10}$ and 8×10^{-13} respectively for points on the surface of the Earth.

The surface obtained when setting $U_E(\mathbf{w}) + (\omega w \sin\theta)^2 = W_0$ in (5), with ω representing the angular velocity of rotation of the Earth and W_0 the gravitational + centrifugal potential on the geoid, differs from the ellipsoid of the Earth model by less than 10 m. Hence an estimate of the accuracy of (5) can be obtained by considering the maximal difference between the geoid and the reference ellipsoid which can amount to ≈ 100 m (Vanicek & Krakiwsky 1986). Therefore expression (5) for the Earth's gravitational potential should not be used if accuracies better than one part in 10^{14} are required.

On the coast the mean sea level can be determined using a tidal gauge. This level differs from the geoid by what is known as Sea Surface Topology (SST) which can amount to ± 0.7 m (Torge 1989). The SST can be determined with an accuracy of 0.1 m (Torge 1989) using oceanographic methods and satellite altimetry which induces an uncertainty of 1×10^{-17} in (4). The uncertainty in the knowledge of the potential on the geoid $\mathbf{W}_0$, which is of the order of $\pm 1 m^2/s^2$ (Bursa 1992, 1993), contributes another part in 10^{17}. The gravitational and centrifugal potential difference between mean sea level and an arbitrary point far from the coast can be obtained by geometrical leveling with simultaneous gravimetric measurements. The accumulated uncertainty when using modern leveling techniques and gravimetry is below 0.5 mm/$\sqrt{km}$ (Kasser 1989) and does therefore not exceed a few centimeters even over large distances. In many countries leveling networks have been established at accuracies of 1–2 mm/$\sqrt{km}$ for primary points, the use of which would again induce errors at the centimetric level.

Therefore the constant part of the total potential at any point on the Earth's surface can be determined with an accuracy better than 2.5 m^2/s^2 using a tidal gauge and good geometrical leveling. The main contributions to this uncertainty are due to inaccuracies in the determination of W and the SST. This limits the calculation of the second term in (4) at the level of $2 \rightarrow 3 \times 1O^{-17}$ which is the limit for syntonization of clocks with respect to coordinate time (TCG or TT) on the surface of the Earth.

Uncertainties in the potential model GEM–T3 (Lerch et al. 1992) and the determination of the satellite orbit (5 cm seems a realistic value) limit the accuracy of syntonization of satellite clocks at a few parts in 10^{18} for low altitudes (semimajor axis < 15000 km). For higher altitudes the effect of these uncertainties is below the 10^{-18} level.

Therefore all terms necessary for the syntonization with respect to TCG of clocks on board high altitude satellites (a > 15000 km) can be calculated to accuracies better than one part in 10^{18}.

3. Transformation to TT

TCG is related to TT by a relativistic transformations, hence any clock that is syntonized with respect to TCG can also be syntonized with respect to TT. In this case the accuracy of syntonization may be limited by the uncertainty in the determination of the parameters participating in the transformation.

The IAU defined TT as a geocentric coordinate time scale differing from TCG by a constant rate, the scale unit of TT being chosen so that it agrees with the SI second on the geoid (IAU 1991). TT is an ideal form of the International Atomic Time TAI, apart from a constant offset. It can be obtained from TCG via the transformation:

$$dTT/dTCG = 1 - L_g \tag{6}$$

with $L_g = W_0/c^2 = 6.9692903 \times lO^{10} \pm 1 \times 1O^{-17}$.

It follows that at present the accuracy of syntonization with respect to TT is limited at the 10^{-17} level due to uncertainties in the determination of the potential on the geoid W_0, even for clocks on board terrestrial satellites.

This limit is inherent to the definition of TT and can therefore only be improved by a reduction of the uncertainty in the determination of W_0. If highly stable clocks on board terrestrial satellites are to be used for the realization of TT at accuracies exceeding this limit it might prove necessary to change the definition. One possibility would be to turn L_g into a defining constant with a fixed value, which would at the same time provide a relativistic definition of the geoid (Bjerhammar 1985, Soffel et al. 1988).

4. Time varying effects

For several applications of highly stable clocks, one is interested in the stability of the relative rate between two clocks, and therefore only time varying effects need to be considered, which can be calculated at the 10^{-18} accuracy level even for clocks on the surface of the Earth. Table II gives a summary of all such effects estimated to exceed the 10^{-18} limit.

Volcanic, coseismic, geodynamic and man-made (e.g. exploitation of oil, gas, geothermal fields) effects are highly localized and only need to be taken into account at some particular locations.

Polar motion and tidal effects are of periodic nature with essentially diurnal and semi-diurnal tidal periods, and the Chandler period (430 days) for the movement of the pole. If the clocks in question are syntonized using repeated time transfers (see (3)) at picosecond accuracy, tidal terms can be neglected as their short periods prevent their amplitudes in the time domain from reaching one picosecond (Klioner 1992).

For atmospheric pressure variations of $\pm$ 10 mbar on a global scale (corresponding to seasonal changes), the effect on the rate of a clock on the Earth's surface can reach $\pm$ 2 parts in 10^{18} with local pressure changes ((anti)cyclones with pressure variations of up to $\pm$ 60 mbar) giving rise to a correction of up to $\pm 2.7 \times 1O^{-18}$ (Rabbel and Zschau 1985).

5. Conclusion

We have presented a theory for the syntonization of clocks with respect to Geocentric Coordinate Time (TCG) including all terms greater than 10^{-18} for clocks on board satellites at altitudes exceeding 15000 km. For this purpose terms of order c^{-3} and c^{-4} in the metric can be neglected, which implies that the specification of coordinate conditions and the state of rotation of the reference system is not necessary.

Syntonization with respect to Terrestial Time (TT), an ideal form of TAI, is limited at the 10^{-17} accuracy level due to the uncertainty in the determination of the potential on the geoid W_0

inherent to its definition.

For clocks on the Earth's surface syntonization with respect to TCG or TT is limited at an accuracy of $2 \rightarrow 3 \times 1O^{-17}$ by uncertainties in the determination of the geopotential at the location of the clock.

We briefly discussed time varying effects that may influence the stability of the relative rate of two clocks. These can be calculated at the 10^{-18} accuracy level even for clocks on the Earth's surface.

At present atomic clocks are approaching stabilities of the order 10^{-18} (Maleki 1993) with further improvements expected in the near future. For comparisons of these highly stable clocks over large distances, and their application in experimental relativity, geodesy, geophysics etc... a sufficiently accurate relativistic theory for their syntonization, like the one presented in this paper, seems indispensable.

Together with a previous paper (Petit & Wolf 1994) the results obtained here amount to a complete relativistic theory for the realization of a geocentric coordinate time scale at a synchronization and syntonization accuracy of one picosecond and 10^{-18} respectively.

References

Allan D.W. & Ashby N., 1986, in: Kovalevsky J., Brumberg V.A. (eds.), Relativity in Celestial Mechanics and Astronomy. Proceedings of the IAU Symposium No. 114 Leningrad 1985, Reidel, Dordrecht

Bjerhammar A., 1985, Bull. Géod. 59, p.207

Brumberg V. A. & Kopejkin, 1990, Celestial Mechanics 48, 23–44.

Bursa M. et. al., 1992,, Studia Geoph. et Geod., 36, 101–114.

Bursa M., 1993, I.A.G. Special Commission SCR – Fundamental Constants (SCFC), Circular No 5.

CCIR, 1990, International Radio Consultative Committee, Reports, Annex to Volume VII, p.150

IAU, 1991, IAU transactions Vol. XXIB, 1991, Proe. 21st Gen. Assembly Buenos Aires, Kluwer Acad. Publ., Dordrecht, Boston, London.

Kasser M., 1989, C. R. Acad. Sci. Paris, t. 309, Sdrie 11, p.695

Klioner S.A., 1992, Celestial Mechanics and Dynamical Astronomy 53, p.81

Kopejkin S.M., 1988, Celestial Mechanics 44, 87–115.

Lerch F.J. et al., 1992, NASA Technical Memorandum 104555.

Maleki L., 1993, Proceedings 25th PTTI Meeting 1993, Nasa Conference Publication 3267, 549–560.

Moritz H., 1961, Osterreichische Zeitschrift far Vermessungswesen, v. 49, 11–15

Petit G. & Wolf P., 1993, Proceedings 25th PTTI meeting, 205–215.

Petit G. & Wolf P., 1994, Astronomy and Astrophysics 286, 971–977.

Rabbel W. & Zschau J., 1985, J. Geophysics 56, p.81

Soffel M. et al., 1988, Manuscripta Geodetica 13, p.143

Torge W., *1989*, Gravimetry, de *Gruyter*, *Berlin*

Vanieek P. & Krakiwsky E., 1986, Geodesy the Concepts, North Holland.

Effect	Order of magnitude	Uncertainty
Earth's grav. pot.	7×10^{-10}	10^{-17}
Centrifugal pot.($v^2/2/c^2$)	1×10^{-12}	$< 10^{-18}$
Volcanic and coseismic (highly localised)	$< 10^{-16}$	
External masses (moon, sun)	10^{-17}	$< 10^{-18}$
Solid Earth tides	10^{-17}	$< 10^{-18}$
Ocean tides	10^{-17}	$< 10^{-18}$

Table Ia: Effects on syntonization with respect to TCG of clocks on the Earth's surface; Orders of magnitude and uncertainties of the corrections.

Effect		Order of magnitude	Uncertainty
Earth's grav. pot.		$< 6 \times 10^{-10}$	few 10^{-18} (GEM-T3) $< 10^{-18}$ at $a>10000$ km few 10^{-18} (5 cm orbit uncertainty) $< 10^{-18}$ at $a>15000$ km
2nd order Doppler ($v^2/2/c^2$)		$< 3 \times 10^{-10}$	$< 10^{-18}$ at $a>15000$ km
External masses:	Moon	4×10^{-13}	$<10^{-18}$
(at a = 300000 km)	Sun	4×10^{-14}	
	Venus	6×10^{-18}	
Solid Earth tides Ocean tides Polar motion Atmospheric pressure		10^{-18} (at low altitudes)	$<10^{-18}$

Table Ib: Effects on syntonization with respect to TCG of clocks on board terrestial satellites; Orders of magnitude and uncertainties of the corrections.

Effect	Order of magnitude	Uncertainty
Volcanic and coseismic (highly localised)	$< 10^{-16}$	
Geodynamic and man-made (localised and long-term > 1 year)	$< 10^{-16}$	
External masses (moon, sun)	10^{-17}	$< 10^{-18}$
Solid Earth tides	10^{-17}	$< 10^{-18}$
Ocean tides	10^{-17}	$< 10^{-18}$
Polar motion (long-term ~ 430 days)	10^{-18}	$< 10^{-18}$
Atmospheric pressure	10^{-18}	$< 10^{-18}$

Table II: Time varying effects on the Earth's surface for the determination of the relative rate of two clocks; Orders of magnitude and uncertainties of the corrections.

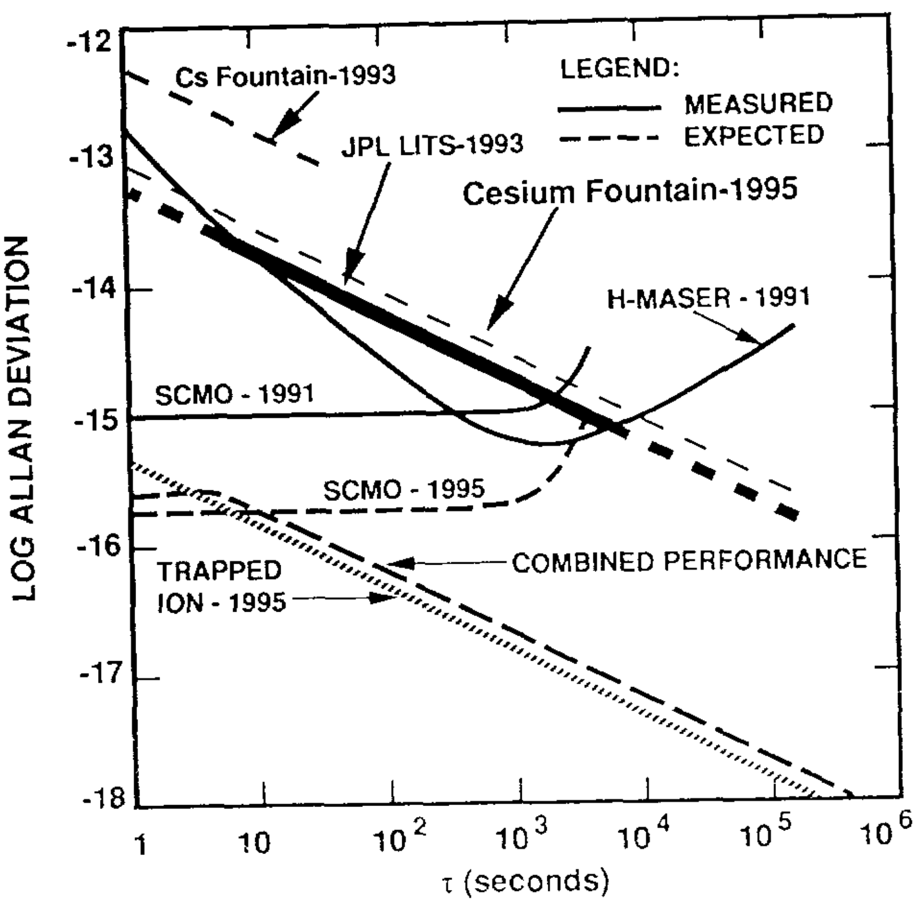

Fig. 1: Present and expected clock stabilities (from Maleki (1993)).

QUESTIONS AND ANSWERS

RICHARD KEATING (USNO): I just have a comment. I don't think that the kind of presentation you just made is particularly useful. I think it's rather misleading. And I would like to say that because about seven years ago, I fired up an old pendulum clock at the request of a retired General Electric engineer. And if I had thought along the lines that you've just presented, I would not have expected to see any effects from, say, a lunar potential. In fact, the pendulum clock is highly sensitive. I could actually tell where the moon was, simply by the effect on the pendulum clock.

These are not relativistic effects, true. But they are far more dramatic, much larger, and they dominate the whole phenomena. So, just to concentrate solely on relativistic effects may be, I think, highly misleading. So, to talk about 10^{-18}, which is a tenth of ps per day, when you actually in reality might have gravitational potential effects, which are the order of ms, I really think you've got bigger problems to worry about. And I think that this kind of paper is misleading.

PETER WOLF (BIPM): Okay, that's your opinion. Fine.

R.J. DOUGLAS (NATIONAL RESEARCH COUNCIL OF CANADA): I would like to come to Peter's defense and say this is one of the most useful kinds of things, because it tells where the limits are. It makes no sense to be thinking about designing optical frequency standards that are going to be useful for time keeping, that are alleged to be possibly stable to parts in 10^{20}. Things that tell you where to stop the development are very useful for systems designers.

GERNOT M. WINKLER (USNO): I would raise the question about the semantics. You are using "syntonization," I believe, in the sense of the ability to absolutely calculate frequency differences. Because, you can always syntonize two standards to each other to see their signals. But you cannot compute the actual frequency difference on an absolute basis.

So, I think there is maybe a need to refine our semantics a little bit.

PETER WOLF (BIPM): I completely agree, yes. There is a big semantic problem concerning the word "syntonization." I have tried to consistently use it in two senses, "syntonization" of two clocks, one relative to another; "syntonization" with respect to coordinate time, which is an entirely different thing.

There might also be several other problems. I do think there's a semantic problem there, but that's only to be solved in time with people getting used to the different things going on.

HENRY FLIEGEL (AEROSPACE CORP.): I want to make one brief comment. I found your paper very useful and interesting. As far as terminology is concerned, I have one brief (almost theological) nit, and that is that I suppose the way to describe the gravitational series, the harmonic expansion, is as very slowly convergent, rather than divergent.

PETER WOLF (BIPM): On the surface of the earth?

HENRY FLIEGEL (AEROSPACE CORP.): I believe so, because if it were divergent, that would mean that we ran eventually into a white noise regime.

PETER WOLF (BIPM): I have a paper which I can show you, which dates back to 1960, which does theoretically prove to show that you cannot be certain that on any point on the surface of the earth this vertical harmonic expansion will be convergent.

HENRY FLIEGEL (AEROSPACE CORP.): In that case, you have refuted all your critics. I would like to see your paper.

PETER WOLF (BIPM): I'm afraid it's in German, Doctor.

HENRY FLIEGEL (AEROSPACE CORP.): Well, I read German, no problem.